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Existence Solution for Fractional Mean-Field Backward Stochastic Differential Equation with Stochastic Linear Growth Coefficients

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Abstract

We deal with fractional mean field backward stochastic differential equations with hurst parameter $H \in (\frac{1}{2}, 1)$ when the coefficient f satisfy a stochastic Lipschitz conditions, we prove the existence and uniqueness of solution and provide a comparison theorem. Via an approximation and comparison theorem, we show the existence of a minimal solution when the drift satisfies a stochastic growth condition.

Keywords: *Fractional Brownian Motion, Backward Stochastic Differential Equations, Stochastic linear Growth, Stochastic Lipschitz-continuous, Itô's Fractional Formula, Comparison Theorem.*

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1 Introduction

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with the covariance function

$$\mathbb{E}\left(B_s^H B_t^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For $H = \frac{1}{2}$, the process B^H is a classical Brownian motion. This process is a self-similar, i.e. B^H at has the same law as B_{at}^H for any $a > 0$. In the case $H > \frac{1}{2}$, the process B^H exhibits long range dependence. These properties make this process a useful driving noise in models arising in finance, physics, telecommunication networks and other fields. However, since B^H with $H > \frac{1}{2}$ is not a semimartingale, we cannot use the classical theory of stochastic calculus to define the fractional stochastic integral. Essentially, two different types of integrals with respect to fBm have been defined and developed. The first one is the pathwise Riemann Stieltjes integral which exists if the integrand has a continuous paths of order $\alpha > 1 - H$ (see Young, [15]). This integral has the properties of Stratonovich integral, which leads to difficulties in the applications. The second one, introduced in [5] is the divergence operator (Skorokhod integral), defined as the adjoint of the derivative operator in the framework of the Malliavin calculus. Since this stochastic integral satisfies the zero mean property and it can be expressed as the limit of Riemann sums defined using Wick products, it was later developed by many authors.

Backward stochastic differential equations (BSDEs in short) driven by Brownian motion were introduced by Bismut [2] for the linear case. In 1990, the nonlinear backward stochastic differential equations were introduced by Pardoux and Peng [12]. Since then, these pioneer works are extensively used in many fields like mathematical finance [6], stochastic optimal control and stochastic games [8]. At the same time, for better applications, BSDE itself has been developed into many different branches. For example, Buckdahn et al.[3, 4] introduced the so-called mean-field BSDEs (MF-BSDEs), owing to the fact that mathematical mean-field approaches have important applications in many domains, such as economics, physics and game theory.

BSDE driven by fractional Brownian motion were introduced by Bender [1] for the linear case. The nonlinear BSDEs with respect to fBm were first studied by Hu [7], Hu and Peng [9], they obtained the existence and uniqueness of the solution but with some restrictive assumption. Then Maticiuc and Nie [11] improved their result and omitted this assumption. They also developed a theory of backward stochastic variational inequalities, i.e. they proved an existence and uniqueness of the solution of reflected BSDEs driven by fBm.

In this paper we study the nonlinear dynamics systems governed by the mean field BSDEs driven by fBm with Hurst parameter $H > \frac{1}{2}$. First, we establish existence and uniqueness of solutions of such an equation under stochastic Lipschitzian condition and establish a comparison theorem. Also by the method developed in [10], we prove that kind of equation has a minimal solution under continuous and stochastic linear growth conditions.

The organization of our paper is as follows: The existence and uniqueness result for the solution of fractional mean-field backward SDE under stochastic Lipschitz condition and comparison theorem are given in section 2. Finally, section 3 is devoted to the existence of minimal solution for fractional mean-field BSDE under continuous and stochastic linear growth.

2 Fractional Mean-Field Backward SDE

In this section, we recall the existence and uniqueness result and the comparison theorem for fractional mean-field BSDEs under stochastic Lipschitz conditions.

2.1 Definitions and Notations

For a fixed real $t \in [0,T]$ and suppose that $B^H = \{B_t^H, t \geq 0\}$ is a one-dimensional fBm defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $\frac{1}{2} < H < 1$ and $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the natural filtration generated by B^H. Assume that

- η_0 is a given constant.
- $b, \ \sigma \colon [0,T] \to \mathbb{R}$ are continuous deterministic functions, σ is differentiable and such that $\sigma(t) \neq 0 \ \forall t \in [0,T]$,

note that, since $||\sigma||_t^2 = H(2H-1)\int_0^t \int_0^t |u-v|^{2H-2} \sigma(u)\sigma(v)\,dudv$, we have $\frac{d}{dt}\left(||\sigma||_t^2\right) = \sigma(t)\hat{\sigma}(t) \geq 0$, where $\hat{\sigma}(t) = \int_0^t \phi(t-v)\sigma(v)\,dv$.

Let $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right) = (\Omega \times \Omega, \mathcal{F}_t \otimes \mathcal{F}_t, \mathbb{P} \otimes \mathbb{P})$ be the (non-completed) product of $(\Omega, \mathcal{F}, \mathbb{P})$ with itself. We denote the filtration of this product space by $\bar{\mathcal{F}} = \left\{\bar{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{F}_t, \ 0 \leq t \leq T\right\}$.

A random variable $\xi \in \mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ originally defined on Ω is extended canonically to $\bar{\Omega}$: $\acute{\xi}(\acute{\omega}, \omega) = \xi(\acute{\omega}), (\acute{\omega}, \omega) \in \bar{\Omega} = \Omega \times \Omega$.

For every $\theta \in \mathbb{L}^1\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right)$, the variable $\theta(\cdot, \omega) : \Omega \to \mathbb{R}$ belongs to $\mathbb{L}^1\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right)$, $\mathbb{P}(d\omega) - a.s,$. We denote its expectation by $\mathbb{E}'(\theta(\cdot,\omega)) = \int_\Omega \theta(\acute{\omega}, \omega)\,\mathbb{P}(d\acute{\omega})$.

Notice that

$$\begin{cases} \mathbb{E}'(\theta) = \mathbb{E}'(\theta(\cdot,\omega)) \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \\ \text{and} \\ \overline{\mathbb{E}}(\theta) = \int_{\bar{\Omega}} \theta d\bar{\mathbb{P}} = \int_\Omega \mathbb{E}'(\theta(\cdot,\omega))\,\mathbb{P}(d\omega) = \mathbb{E}(\mathbb{E}'(\theta)). \end{cases}$$

Let η_t be a solution of the following SDE with respect to fractional Brownian motion

$$\begin{cases} d\eta_t = b(t)\,dt + \sigma(t)\,dB_t^H, \\ \eta_0 = x_0. \end{cases} \tag{1}$$

Let $f : \Omega \times [0,T] \times \mathbb{R}^5 \longmapsto \mathbb{R}$ be measurable functions.

Now we consider the following mean field backward SDE with respect to fBm:

$$\begin{cases} dY_t = -\mathbb{E}'\left(f(t, \eta_t, Y_t, Z_t, \acute{Y}_t, \acute{Z}_t)\right)dt + Z_t dB_t^H, \\ Y_T = \xi \end{cases} \tag{2}$$

Before giving the definition of solutions of BSDE (2), we introduce for a fixed $\delta > 0$ the following sets:

- $\mathbb{L}^2(\delta, \mathcal{F}_T, \mathbb{R})$ is the space of $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variables such that
$$\mathbb{E}\left(e^{\delta A(T)}|\xi|^2\right) < \infty.$$
- $\mathcal{C}_{pol}^{1,2}([0,T] \times \mathbb{R})$ is the space of all $\mathcal{C}^{1,2}$-functions over $[0,T] \times \mathbb{R}$, which together with their derivatives are of polynomial growth.
- $\mathcal{V}_{[0,T]} = \left\{Y = \psi(\cdot, \eta);\ \psi \in \mathcal{C}_{pol}^{1,2}([0,T] \times \mathbb{R})\right\}$, $\frac{d\psi}{dt}$ is bounded, $t \in [0,T]$.
- $\tilde{\mathcal{V}}_{[0,T]}^{H,a}$ and $\tilde{\mathcal{V}}_{[0,T]}^{H}$ denote the completion of $\mathcal{V}_{[0,T]}$ under the following norm, respectively:
$$||Y||_v^a \triangleq \left(\mathbb{E}\int_0^T a^2(t)\,t^{2H-1}e^{\delta A(t)}|Y_t|^2\,dt\right)^{\frac{1}{2}},$$
$$||Z||_v \triangleq \left(\mathbb{E}\int_0^T t^{2H-1}e^{\delta A(t)}|Z_t|^2\,dt\right)^{\frac{1}{2}},$$
where $\delta > 0$ is a constant. It is easy to see that $\tilde{\mathcal{V}}_{[0,T]}^{H} \subset \tilde{\mathcal{V}}_{[0,T]}^{H,a} \subset \mathbb{L}^2(0,T,\mathbb{R})$.

Definition. A solution of equation (2) is a pair of processes (Y,Z) which belongs to the space $\tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a} \times \tilde{\mathcal{V}}_{[0,T]}^{H}$ and satisfies (2).

The setting of our problem is to find a pair of processes $(Y_., Z_.) \in \tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a} \times \tilde{\mathcal{V}}_{[0,T]}^{H}$ satisfying the BSDE (2). In the following, we will prove the existence and uniqueness of Eq. (2).

2.2 Fractional Mean-Field BSDE with Stochastic Lipschitz Coefficients.

Assume the coefficient $f \colon \Omega \times [0,T] \times \mathbb{R}^5 \to \mathbb{R}$ and the terminal value $\xi \colon \Omega \to \mathbb{R}$ satisfy the following assumptions, for $\delta > 0$:

We say that the coefficient f satisfies assumptions **(H1)** if the following holds:

(H1.1) There exist two non-negative processes $\{\alpha(t)\}_{0 \leq t \leq T}$ and $\{\beta(t)\}_{0 \leq t \leq T}$ such that:

1. For any $0 \leq t \leq T$, $\alpha(t)$ and $\beta(t)$ are $\mathcal{F}_t$-measurable.
2. For all $0 \leq t \leq T$ $x \in \mathbb{R}$, $(y, \acute{y}) \in \mathbb{R}^2$, $\left(\tilde{y}, \tilde{y}'\right) \in \mathbb{R}^2$, $(z, \acute{z}) \in \mathbb{R}^2$ and $\left(\tilde{z}, \tilde{z}'\right) \in \mathbb{R}^2$, we have
$$\begin{cases} \left|f(t,\omega,x,y,z,\tilde{y},\tilde{z}) - f(t,\omega,x,\acute{y},\acute{z},\tilde{y}',\tilde{z}')\right| \\ \leq \alpha(t)\left(|y-\acute{y}| + \left|\tilde{y}-\tilde{y}'\right|\right) + \beta(t)\left(|z-\acute{z}| + \left|\tilde{z}-\tilde{z}'\right|\right). \end{cases}$$

(H1.2) For all $0 \leq t \leq T$, $a^2(t) = \alpha(t) + \beta^2(t) > 0$, and $A(t) = \int_0^t a^2(s)\,ds < \infty$.

(H1.3) The integrability condition holds:
$$\mathbb{E}\left(\int_0^T e^{\delta A(t)}\frac{|f(t,\omega,x,0,0,0,0)|^2}{a^2(t)}dt\right) < \infty.$$

(H1.4) $\xi \in \mathbb{L}^2(\delta, \mathcal{F}_T, \mathbb{R})$.

To solve equation (2), we investigate first the case, where the generator does not depend on the unknown processes Y and Z. Namely, we consider the stochastic equation

$$Y_t = \xi + \int_t^T \mathbb{E}'(f(s,\eta_s))\,ds - \int_t^T Z_s dB_s^H, \ 0 \leq t \leq T, \tag{3}$$

where $f(t,\omega,\eta_t) \in \mathbb{L}^2(\delta, \mathcal{F}_t, \mathbb{R})$ satisfies the following integrability condition:

(H1.3)$'$ $\mathbb{E}\left(\int_0^T e^{\delta A(t)}\frac{|f(t,\omega,\eta_t)|^2}{a^2(t)}dt\right) < \infty.$

Proposition. For any $\delta > 0$, there exists a unique solution $(Y, Z) \in \tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}$ to equation (3). Moreover, there exists a positive constant $C_{\delta,M}$ depending on δ and M such that for any $0 \leq t \leq T$,

$$\begin{aligned} &\mathbb{E}e^{\delta A(t)} |Y_t|^2 + \mathbb{E}\int_t^T e^{\delta A(s)} a^2(s) |Y_s|^2 \\ &+\mathbb{E}\int_t^T e^{\delta A(s)} s^{2H-1} |Z_s|^2 ds \\ \leq\ & C_{\delta,M}\theta(\xi, t, T), \end{aligned} \tag{4}$$

where
$\theta(\xi, t, T) = \mathbb{E}\left(e^{\delta A(T)} |\xi|^2 + \frac{2}{\delta}\int_t^T e^{\delta A(s)} \frac{|f(s,\eta_s)|^2}{a^2(s)} ds\right)$.

Proof. Note that by [9], the stochastic equation (3) admits a unique solution

$$(Y, Z) \in \tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}.$$

It remains to show that $Y \in \tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]}$.

Itô's formula applied to equation (3) yields for $0 \leq t \leq T$ and $\delta > 0$,

$$\begin{aligned} &e^{\delta A(t)} |Y_t|^2 + \delta\int_t^T a^2(s) e^{\delta A(s)} |Y_s|^2 ds \\ =\ & e^{\delta A(T)} |\xi|^2 + 2\int_t^T e^{\delta A(s)} Y_s \mathbb{E}'(f(s, \eta_s)) ds \\ &-2\int_t^T e^{\delta A(s)} Y_s Z_s dB^H_s - 2\int_t^T e^{\delta A(s)} \mathcal{D}^{\mathcal{H}}_s Y_s Z_s ds. \end{aligned}$$

Taking expectation, we get

$$\begin{aligned} &\mathbb{E}\left(e^{\delta A(t)} |Y_t|^2 + \delta\int_t^T a^2(s) e^{\delta A(s)} |Y_s|^2 ds\right) \\ =\ & \mathbb{E}\left(e^{\delta A(T)} |\xi|^2 + 2\int_t^T e^{\delta A(s)} Y_s f(s, \eta_s) ds\right) \\ &-2\mathbb{E}\left(\int_t^T e^{\delta A(s)} \mathcal{D}^{\mathcal{H}}_s Y_s Z_s ds\right). \end{aligned} \tag{5}$$

Then using the inequality $2ab \leq a^2\epsilon + \frac{b^2}{\epsilon}$, we have

$$2Y_s f(s, \eta_s) \leq \frac{\delta}{2} a^2(s) |Y_s|^2 + \frac{2}{\delta}\frac{|f(s, \eta_s)|^2}{a^2(s)}.$$

It is known (see for example Hu and Peng, [9], Maticiuc and Nie, [11]) that $\mathcal{D}^{\mathcal{H}}_t Y_t = (\hat{\sigma}(t)/\sigma(t)) Z_t$. Moreover by Remark 6 in Maticiuc and Nie [11], there exists $M > 0$ such that for all $t \in [0, T]$, $t^{2H-1}/M \leq \hat{\sigma}(t)/\sigma(t) \leq Mt^{2H-1}$.

$$\begin{aligned} &\mathbb{E}e^{\delta A(t)} |Y_t|^2 + \frac{\delta}{2}\mathbb{E}\int_t^T a^2(s) e^{\delta A(s)} |Y_s|^2 ds \\ &+\frac{2}{M}\mathbb{E}\int_t^T e^{\delta A(s)} s^{2H-1} |Z_s|^2 ds \\ \leq\ & \theta(\xi, t, T). \end{aligned} \tag{6}$$

Choosing δ, $M > 2$, we deduce from (6),

$$\begin{aligned} &\mathbb{E}e^{\delta A(t)} |Y_t|^2 + \mathbb{E}\int_t^T a^2(s) e^{\delta A(s)} |Y_s|^2 ds \\ &+\mathbb{E}\int_t^T e^{\delta A(s)} s^{2H-1} |Z_s|^2 ds \\ \leq\ & C_{\delta,M}\theta(\xi, t, T). \end{aligned}$$

This implies in particular that $(Y, Z) \in \tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}$ and (4) follows. □

Theorem. *Assume that assumptions* ***(H1)*** *hold. Then, for δ sufficiently large, the fractional MF-BSDE (2) has a unique solution $(Y, Z) \in \tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}$.*

Proof. Existence part: We consider the sequence $(Y^n, Z^n)_{n\geq 0}$ defined by

$$\begin{cases} -dY^{n+1}_t = \mathbb{E}'\left(f(t, \eta_t, Y^n_t, Z^n_t, \acute{Y}^n_t, \acute{Z}^n_t)\right) dt - Z^{n+1}_t dB^H_t \\ Y^{n+1}_T = \xi, \quad 0 \leq t \leq T. \end{cases} \tag{7}$$

Since for a fixed $n \in N$, the coefficient f of the fractional MF-BSDE (3.7) does not depend on the solution $\left(Y^{n+1}, Z^{n+1}\right)$ it follows from the previous proposition that the sequence $(Y^n, Z^n)_{n\geq 0}$ is well defined in $\tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}$.

Our strategy consists of proving that $(Y^n, Z^n)_{n\geq 0}$ is a Cauchy sequence. To this, let $n \geq 1$ and define for a process $\pi \in \{Y, Z\}$, $\Delta\pi^{n+1} = \pi^{n+1} - \pi^n$ and

$$\begin{aligned} \Delta f^{n+1}(s, \eta_s) =\ & \mathbb{E}' f(s, \eta_s, Y^{n+1}_s, Z^{n+1}_s, \acute{Y}^{n+1}_s, \acute{Z}^{n+1}_s) \\ &-\mathbb{E}' f(s, \eta_s, Y^n_s, Z^n_s, \acute{Y}^n_s, \acute{Z}^n_s). \end{aligned}$$

It is readily seen that the pair $\left(Y^{n+1}_t, Z^{n+1}_t\right)_{t\in[0,T]}$ solves the following fractional MF-BSDE:

$$\Delta Y^{n+1}_t = \int_t^T \Delta f^n(s, \eta_s) ds - \int_t^T \Delta Z^{n+1}_s dB^H_s, \ 0 \leq t \leq T. \tag{8}$$

Itô's formula, applied to $e^{\delta A(t)} \left|\Delta Y^{n+1}_t\right|^2$, yields for $0 \leq t \leq T$ and $\delta > 0$,

$$\begin{aligned} &\mathbb{E}e^{\delta A(t)} \left|\Delta Y^{n+1}_t\right|^2 + \delta\mathbb{E}\int_t^T a^2(s) e^{\delta A(s)} \left|\Delta Y^{n+1}_s\right|^2 ds \\ &+2\mathbb{E}\int_t^T e^{\delta A(s)} \mathcal{D}^{\mathcal{H}}_s \Delta Y^{n+1}_s \Delta Z^{n+1}_s ds \\ =\ & 2\mathbb{E}\int_t^T e^{\delta A(s)} \Delta Y^{n+1}_s \Delta f^n(s, \eta_s) ds. \end{aligned}$$

It is known (see for example Hu and Peng, [9], Maticiuc and Nie, [11]) that $\mathcal{D}^{\mathcal{H}}_t \Delta Y^{n+1}_t = (\hat{\sigma}(t)/\sigma(t)) \Delta Z^{n+1}_t$. Moreover by Remark 6 in Maticiuc and Nie [11], there exists $M > 0$ such that for all $t \in [0, T]$, $t^{2H-1}/M \leq \hat{\sigma}(t)/\sigma(t) \leq Mt^{2H-1}$.

$$\begin{aligned} &\mathbb{E}e^{\delta A(t)} \left|\Delta Y^{n+1}_t\right|^2 + \delta\mathbb{E}\int_t^T a^2(s) e^{\delta A(s)} \left|\Delta Y^{n+1}_s\right|^2 ds \\ &+\frac{2}{M}\mathbb{E}\int_t^T s^{2H-1} e^{\delta A(s)} |\Delta Z^n_s|^2 ds \\ =\ & 2\mathbb{E}\int_t^T e^{\delta A(s)} \Delta Y^{n+1}_s \Delta f^n(s, \eta_s) ds. \end{aligned} \tag{9}$$

By assumption $(H1.1)\,(2)$, we deduce that

$$\begin{aligned} & 2\mathbb{E}\int_t^T e^{\delta A(s)}\Delta Y_s^{n+1}\Delta f^n(s,\eta_s)ds \\ \leq\ & 2\mathbb{E}\int_t^T e^{\delta A(s)}\left|\Delta Y_s^{n+1}\right|\left|\Delta f^n(s,\eta_s)\right|ds, \\ \leq\ & 2\mathbb{E}\int_t^T e^{\delta A(s)}\alpha(s)\left|\Delta Y_s^{n+1}\right|\left|\Delta Y_s^n\right|ds \\ & +2\mathbb{E}\int_t^T e^{\delta A(s)}\alpha(s)\left|\Delta Y_s^{n+1}\right|\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)ds \quad (10) \\ & +2\mathbb{E}\int_t^T e^{\delta A(s)}\beta(s)\left|\Delta Y_s^{n+1}\right|\left|\Delta Z_s^n\right|ds \\ & +2\mathbb{E}\int_t^T e^{\delta A(s)}\beta(s)\left|\Delta Y_s^{n+1}\right|\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)ds. \end{aligned}$$

Therefore by choosing $\delta \geq 1$, using Hölder's inequality and Jensen's inequality we get

$$\begin{aligned} & \mathbb{E}e^{\delta A(t)}\left|\Delta Y_t^{n+1}\right|^2+\delta\mathbb{E}\int_t^T a^2(s)\,e^{\delta A(s)}\left|\Delta Y_s^{n+1}\right|^2ds \\ & +\frac{2}{M}\mathbb{E}\int_t^T s^{2H-1}e^{\delta A(s)}\left|\Delta Z_s^n\right|^2ds \\ \leq\ & 4\mathbb{E}\int_t^T a^2(s)\left\{\left(e^{\delta A(s)}\mathbb{E}\left|\Delta Y_s^{n+1}\right|^2\right)^{\frac{1}{2}}\right. \\ & \left.\times\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2\right)^{\frac{1}{2}}\right\}ds \qquad (11) \\ & +4\mathbb{E}\int_t^T\left(\beta^2(s)\,e^{\delta A(s)}\mathbb{E}\left|\Delta Y_s^{n+1}\right|^2\right)^{\frac{1}{2}} \\ & \times\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2\right)^{\frac{1}{2}}ds. \end{aligned}$$

Denote $x(s)=\left(e^{\delta A(s)}\mathbb{E}\left|\Delta Y_s^{n+1}\right|^2\right)^{\frac{1}{2}}$, Then from (3.10)

$$\begin{aligned} x(t)^2 \leq\ & 4\int_t^T a^2(s)\,x(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2\right)^{\frac{1}{2}}ds \\ & +4\int_t^T\beta(s)\,x(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2\right)^{\frac{1}{2}}ds. \end{aligned}$$

Using Lemma 20 in Maticiuc and Nie [11] above inequality, it follows that

$$\begin{aligned} x(t) \leq\ & 4\int_t^T a^2(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2\right)^{\frac{1}{2}}ds \\ & +4\int_t^T\beta(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2\right)^{\frac{1}{2}}ds. \end{aligned}$$

Therefore for $t \in [t_k, T]$

$$\begin{aligned} x(t)^2 \leq\ & 32\left(\int_t^T a^2(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2\right)^{\frac{1}{2}}ds\right)^2 \\ & +32\left(\int_t^T\beta(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2\right)^{\frac{1}{2}}ds\right)^2. \end{aligned}$$

Now we compute

$$\begin{aligned} & \int_{t_k}^T x(s)^2\,ds \\ \leq\ & 32(T-t_k)\left(\int_{t_k}^T a^2(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2\right)^{\frac{1}{2}}ds\right)^2 \quad (12) \\ & +32(T-t_k)\left(\int_{t_k}^T\beta(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2\right)^{\frac{1}{2}}ds\right)^2, \\ =\ & :\Gamma. \end{aligned}$$

For the term Γ in (11)

$$\begin{aligned} & \left(\int_{t_k}^T a^2(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2\right)^{\frac{1}{2}}ds\right)^2 \\ \leq\ & \int_{t_k}^T a^2(s)\,ds\cdot\int_{t_k}^T a^2(s)\,e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2ds, \quad (13) \end{aligned}$$

and

$$\begin{aligned} & \left(\int_{t_k}^T\beta(s)\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2\right)^{\frac{1}{2}}ds\right)^2 \\ =\ & \left(\int_{t_k}^T\frac{\beta(s)}{\sqrt{s^{2H-1}}}\sqrt{s^{2H-1}}\left(e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2\right)^{\frac{1}{2}}ds\right)^2, \\ \leq\ & \int_{t_k}^T\frac{a^2(s)}{s^{2H-1}}ds\cdot\int_{t_k}^T s^{2H-1}e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2ds. \quad (14) \end{aligned}$$

Combining (12) and (13), it follows that

$$\begin{aligned} & \int_{t_k}^T x(s)^2\,ds \\ \leq\ & F\int_{t_k}^T a^2(s)\,e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2ds \qquad (15) \\ & +G\int_{t_k}^T s^{2H-1}e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2ds, \end{aligned}$$

where $F=32(T-t_k)\int_{t_k}^T a^2(s)\,ds<\infty$ and $G=32(T-t_k)\int_{t_k}^T\frac{a^2(s)}{s^{2H-1}}ds<\infty$. And similarly

$$\begin{aligned} & \int_{t_k}^T\frac{1}{s^{2H-1}}x(s)^2\,ds \\ \leq\ & \tilde{F}\int_{t_k}^T a^2(s)\,e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2ds \qquad (16) \\ & +\tilde{G}\int_{t_k}^T s^{2H-1}e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2ds, \end{aligned}$$

where $\tilde{F}=32\left(\frac{T^{2-2H}-t_k^{2-2H}}{2-2H}\right)\int_{t_k}^T a^2(s)\,ds<\infty$ and $\tilde{G}=32\left(\frac{T^{2-2H}-t_k^{2-2H}}{2-2H}\right)\int_{t_k}^T\frac{a^2(s)}{s^{2H-1}}ds<\infty$.

Now from (9) and (10)

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|\Delta Y_t^{n+1}\right|^2 \\
&+\mathbb{E}\int_t^T \frac{\left(\delta^2-2\right)s^{2H-1}-2}{\delta s^{2H-1}}a^2(s) \\
&\times e^{\delta A(s)}\left|\Delta Y_s^{n+1}\right|^2 ds \qquad (17)\\
&+\frac{2}{M}\mathbb{E}\left(\int_t^T s^{2H-1}e^{\delta A(s)}\left|\Delta Z_s^{n+1}\right|^2 ds\right) \\
\leq\ & 2\delta\mathbb{E}\int_t^T e^{\delta A(s)} \\
&\times\left(a^2(s)\left|\Delta Y_s^n\right|^2 ds+s^{2H-1}\left|\Delta Z_s^n\right|^2\right)ds.
\end{aligned}$$

Choosing $\delta>0$ such that $\left(\delta-\frac{2}{\delta}-\frac{2}{\delta s^{2H-1}}\right)>1$ for any $t\leq s\leq T$ and using the inequality (14), and note that $M>2$, we have

$$\begin{aligned}
&\mathbb{E}\int_{t_k}^T e^{\delta A(s)}\left(a^2(s)\left|\Delta Y_s^{n+1}\right|^2+s^{2H-1}\left|\Delta Z_s^{n+1}\right|^2\right)ds \\
\leq\ & \delta MF\mathbb{E}\int_{t_k}^T a^2(s)e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2 ds \\
&+\delta(MG+2)\mathbb{E}\int_{t_k}^T s^{2H-1}e^{\delta A(s)}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2 ds.
\end{aligned}$$

Now choosing $\delta>0$ and taking k large enough such that $\delta MF\leq\frac{1}{4}$ and $\delta(MG+2)\leq\frac{1}{4}$, we deduce

$$\begin{aligned}
&\mathbb{E}\int_{t_k}^T e^{\delta A(s)}a^2(s)\left|\Delta Y_s^{n+1}\right|^2 ds \\
&+\mathbb{E}\int_{t_k}^T e^{\delta A(s)}s^{2H-1}\left|\Delta Z_s^{n+1}\right|^2 ds \\
\leq\ & \frac{1}{2}\mathbb{E}\int_{t_k}^T e^{\delta A(s)}a^2(s)\mathbb{E}\left(\left|\Delta Y_s^n\right|\right)^2 ds \\
&+\frac{1}{2}\mathbb{E}\int_{t_k}^T e^{\delta A(s)}s^{2H-1}\mathbb{E}\left(\left|\Delta Z_s^n\right|\right)^2 ds.
\end{aligned}$$

As a consequence, we deduce that $(Y^n,Z^n)_{n\geq0}$ is a Cauchy sequence in $\tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a}\times\tilde{\mathcal{V}}_{[0,T]}^H$. Then there exists a pair $(Y,Z)\in\tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a}\times\tilde{\mathcal{V}}_{[0,T]}^H$ being a limit of $(Y^n,Z^n)_{n\geq1}$, i.e.

$$\begin{aligned}
&\mathbb{E}\int_0^T a^2(s)e^{\delta A(s)}\left|Y_s^n-Y_s\right|^2 ds \\
&+\mathbb{E}\int_0^T s^{2H-1}e^{\delta A(s)}\left|Z_s^n-Z_s\right|^2 ds \\
\to\ & 0,\ \text{as } n\to\infty.
\end{aligned}$$

It remains to show that the pair (Y,Z) satisfies equation (2) on the interval $[0,T]$. We have for any $t\in[t_k,T]$,

$$\begin{aligned}
&Y_t^{n+1}-\xi-\int_t^T f(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n,\acute{Z}_s^n)ds \\
\underset{n\to\infty}{\to}\ & Y_t-\xi-\int_t^T f(s,\eta_s,Y_s,Z_s,\acute{Y}_s,\acute{Z}_s)ds,
\end{aligned}$$

in $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P})$. And $Z_t^n\mathbf{1}_{[t,T]}\to Z_t\mathbf{1}_{[t,T]}$ in $\mathbb{L}^2(\Omega,\mathcal{F},\mathcal{H})$. Arguing as in the proof of Theorem 23 in Maticiuc and Nie [11] we show that (Y,Z) satisfies (2) on $[t_k,T]$. The next step is to solve the equation on $[t_{k-1},t_k]$. With the same arguments, repeating the above technique we obtain a uniqueness of the solution of MF-BSDE with respect to fBm on the whole interval $[0,T]$.

Uniqueness part: Let $\left(Y_\cdot^1,Z_\cdot^1\right)$ and $\left(Y_\cdot^2,Z_\cdot^2\right)$ two solutions of fractional MF-BSDEs (2), then by Itô's formula applied to $e^{\delta A(t)}\left|Y_t^1-Y_t^2\right|^2$ it follows that, $\forall t\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|Y_t^1-Y_t^2\right|^2+\delta\mathbb{E}\int_t^T a^2(s)e^{\delta A(s)}\left|Y_s^1-Y_s^2\right|^2 ds \\
=\ & 2\mathbb{E}\int_t^T e^{\delta A(s)}\left(Y_s^1-Y_s^2\right) \\
&\times\left(f(s,\eta_s,Y_s^1,Z_s^1,\acute{Y}_s^1,\acute{Z}_s^1)-f(s,\eta_s,Y_s^2,Z_s^2,\acute{Y}_s^2,\acute{Z}_s^2)\right)ds \\
&-2\mathbb{E}\left(\int_t^T e^{\delta A(s)}\mathcal{D}_s^H\left(Y_s^1-Y_s^2\right)\left(Z_s^1-Z_s^2\right)ds\right),
\end{aligned}$$

then, we can write

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|Y_t^1-Y_t^2\right|^2 \\
&+\mathbb{E}\int_t^T\left(\delta-4-\frac{2\delta M}{s^{2H-1}}\right)a^2(s)e^{\delta A(s)}\left|Y_s^1-Y_s^2\right|^2 ds \\
&+\frac{2\delta-2}{\delta M}\mathbb{E}\left(\int_t^T e^{\delta A(s)}s^{2H-1}\left|Z_s^1-Z_s^2\right|^2 ds\right) \\
\leq\ & 0,
\end{aligned}$$

which can be chosen δ, M such that $\delta-4-\frac{2\delta M}{s^{2H-1}}>0$ for any $t\leq s\leq T$ and $\frac{2\delta-2}{\delta M}>0$. Thus, we deduce that

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|Y_t^1-Y_t^2\right|^2 \\
&+\mathbb{E}\int_t^T e^{\delta A(s)}\left(a^2(s)\left|Y_s^1-Y_s^2\right|^2+s^{2H-1}\left|Z_s^1-Z_s^2\right|^2\right)ds \\
\leq\ & 0.
\end{aligned}$$

This implies $Y_t^1=Y_t^2$ and $Z_s^1=Z_s^2$. The result follows. □

2.3 Comparison Theorem

In this subsection we study a comparison theorem for the fractional MF-BSDEs of the following form:

$$\left\{\begin{array}{l}
-dY_t^i=\mathbb{E}'\left(f^i(t,\eta_t,Y_t^i,Z_t^i,\acute{Y}_t^i,\acute{Z}_t^i)\right)dt-Z_t^i dB_t^H \\
Y_T^i=\xi^i,\quad 0\leq t\leq T.
\end{array}\right. \tag{18}$$

where for any $i\in\{1,2\}$, $f^i:\Omega\times[0,T]\times\mathbb{R}^5\to\mathbb{R}$.

We assume in addition that

$$\textbf{(H1.5)}\left\{\begin{array}{l}
\xi^1\leq\xi^2, \\
f^1(s,\eta,y,z,\acute{y},\acute{z})\leq f^2(s,\eta,y,z,\acute{y},\acute{z}), \\
\forall(s,\ \eta,\ y,\ z,\ \acute{y},\ \acute{z})\in[0,T]\times\mathbb{R}^5.
\end{array}\right.$$

We have the following theorem:

Theorem. *Suppose that $\left(\xi^1,f^1\right)$ and $\left(\xi^2,f^2\right)$ satisfy **(H1.1)–(H1.5)**.*

If $\left(Y_s^i,Z_s^i\right)$, $i=1,2$ are solutions to Eq. (18), then we have

$$\forall t\in[0,T],\qquad Y^1\leq Y^2,\qquad \mathbb{P}-a.s.$$

Proof. Let us define $\Delta Y_t = Y_t^2 - Y_t^1$, $\Delta Z_t = Z_t^2 - Z_t^1$, $\Delta \acute{Y}_t = \acute{Y}_t^2 - \acute{Y}_t^1$, $\Delta \acute{Z}_t = \acute{Z}_t^2 - \acute{Z}_t^1$, $\Delta\xi = \xi^2 - \xi^1$ and

$$\begin{aligned} & \Delta f\left(t, \eta_t, \Delta Y_t, \Delta Z_t, \Delta \acute{Y}_t, \Delta \acute{Z}_t\right) \\ = \; & \mathbb{E}^{'} f^2\left(t, \eta_t, Y_t^2, Z_t^2, \acute{Y}_t^2, \acute{Z}_t^2\right) \\ & - \mathbb{E}^{'} f^1(t, \eta_t, Y_t^1, Z_t^1, \acute{Y}_t^1, \acute{Z}_t^1). \end{aligned}$$

It follows that $(\Delta Y_t, \Delta Z_t)_{t\in[0,T]}$ satisfies the fractional MF-BSDE for any $0 \le t \le T$

$$\begin{aligned} & \Delta Y_t \\ = \; & \Delta\xi + \int_t^T \Delta f\left(s, \eta_s, \Delta Y_s, \Delta Z_s, \Delta \acute{Y}_s, \Delta \acute{Z}_s\right) ds \\ & - \int_t^T \Delta Z_s dB_s^H. \end{aligned}$$

Applying Itô's formula to $e^{\delta A(t)} \left|\Delta Y_t^-\right|^2$, we obtain

$$\begin{aligned} & \mathbb{E} e^{\delta A(t)} \left|\Delta Y_t^-\right|^2 + \delta \mathbb{E} \int_t^T a^2(s) e^{\delta A(s)} \left|\Delta Y_s^-\right|^2 ds \\ & + \frac{2}{M} \mathbb{E} \int_t^T \mathbf{1}_{\{\Delta Y_s < 0\}} e^{\delta A(s)} s^{2H-1} \left|\Delta Z_s\right|^2 ds \\ = \; & \mathbb{E}\left(e^{\delta A(T)} \Delta\xi^-\right) - 2\mathbb{E} \int_t^T \mathbf{1}_{\{\Delta Y_s < 0\}} e^{\delta A(s)} \Delta Y_s^- \times \\ & \Delta f\left(s, \eta_s, \Delta Y_s, \Delta Z_s, \Delta \acute{Y}_s, \Delta \acute{Z}_s\right) ds. \end{aligned}$$

Since $\mathbb{E}^{'} f^2\left(t, \eta_t, Y_t^2, Z_t^2, \acute{Y}_t^2, \acute{Z}_t^2\right) \ge \mathbb{E}^{'} f^1(t, \eta_t, Y_t^2, Z_t^2, \acute{Y}_t^2, \acute{Z}_t^2)$ and $\Delta\xi = \xi^1 - \xi^2 \ge 0$, we have

$$\begin{aligned} & \mathbb{E} e^{\delta A(t)} \left|\Delta Y_t^-\right|^2 + \delta \mathbb{E} \int_t^T a^2(s) e^{\delta A(s)} \left|\Delta Y_s^-\right|^2 ds \\ & + \frac{2}{M} \mathbb{E} \int_t^T \mathbf{1}_{\{\Delta Y_s < 0\}} e^{\delta A(s)} s^{2H-1} \left|\Delta Z_s\right|^2 ds \\ \le \; & 2\mathbb{E} \int_t^T \mathbf{1}_{\{\Delta Y_s < 0\}} e^{\delta A(s)} \Delta Y_s^- \\ & \times (\mathbb{E}^{'} f^1\left(s, \eta_s, Y_s^2, Z_s^2, \acute{Y}_s^2, \acute{Z}_s^2\right) \\ & - \mathbb{E}^{'} f^1(s, \eta_s, Y_s^1, Z_s^1, \acute{Y}_s^1, \acute{Z}_s^1)) ds. \end{aligned}$$

From $(H1.1)$, $(H1.2)$ and Young's inequality, we have

$$\begin{aligned} & \Delta Y_s^- \mathbb{E}^{'}\left(f^1\left(s, \eta_s, Y_s^2, Z_s^2, \acute{Y}_s^2, \acute{Z}_s^2\right)\right. \\ & \left. - f^1(s, \eta_s, Y_s^1, Z_s^1, \acute{Y}_s^1, \acute{Z}_s^1)\right) \\ \le \; & \frac{1}{2}\left(4 + \frac{4M}{s^{2H-1}}\right) a^2(s) \left|\Delta Y_s^-\right|^2 + \frac{s^{2H-1}}{2M} \left|\Delta Z_s\right|^2. \end{aligned}$$

Finally, it follows that

$$\begin{aligned} & \mathbb{E}\left(e^{\delta A(t)} \left|\Delta Y_t^-\right|^2 + \int_t^T \left(\delta - 4 - \tfrac{4M}{s^{2H-1}}\right) a^2(s) e^{\delta A(s)} \left|\Delta Y_s^-\right|^2 ds\right) \\ & + \tfrac{1}{M} \mathbb{E}\left(\int_t^T \mathbf{1}_{\{\Delta Y_s < 0\}} e^{\delta A(s)} s^{2H-1} \left|\Delta Z_s\right|^2 ds\right) \le 0 \end{aligned}$$

Therefore, choosing $\delta > 0$ and $M > 0$, such that $\left(\delta - 4 - \frac{4M}{s^{2H-1}}\right) \ge 0$, we derive that $\Delta Y_t^- = 0$ $\mathbb{P} - a.s.$ for all $t \in [0, T]$, which implies that $\Delta Y_t = Y_t^2 - Y_t^1 \ge 0$ $\mathbb{P} - a.s.$ for all $t \in [0, T]$. ☐

3 Fractional MF-BSDE with Continuous and Stochastic Linear Growth Coefficients.

The objective of this section is to prove an existence theorem for MF-BSDEs (2) with Hurst parameter $H > \frac{1}{2}$ when the coefficient f is continuous with stochastic linear growth. More precisely, the coefficient $f : \Omega \times [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ is measurable and the terminal value $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_T-$measurable satisfying the following assumptions for $\delta > 0$.

(A1) The following hold:

i) For fixed ω and t, $f(t, \omega, x, \cdot, \cdot, \cdot)$ is continuous.

ii) For all $(t, \omega, x, y, z, \acute{y}) \in [0,T] \times \Omega \times \mathbb{R}^4$

$$|f(t, \omega, x, y, z, \acute{y})| \le \varphi(t) + r(t)(|x| + |y| + |\acute{y}|) + \theta(t)(|z|).$$

where φ, r and θ are three nonnegative processes such that for $a.e.$ $t \in [0,T]$, $\varphi(t)$, $r(t)$ and $\theta(t)$ $\mathcal{F}_t-$measurable.

iii) For all $(t, \omega, x, y, z, \acute{y}) \in [0,T] \times \Omega \times \mathbb{R}^4$, $f(t, \omega, x, y, z, \acute{y})$ is $\mathcal{F}_t-$measurable.

(A2) For any $t \in [0,T]$,

$$\begin{aligned} a^2(t) &= r(t) + \theta^2(t) > 0, \text{ and} \\ A(t) &= \int_0^t a^2(s) ds < \infty. \end{aligned}$$

(A3) One has

$$\mathbb{E} e^{\delta A(T)} |\xi|^2 + \mathbb{E} \int_0^T e^{\delta A(t)} \left(\frac{|\varphi(t)|^2}{a^2(t)} + a^2(t) |\eta_t|^2\right) dt < \infty.$$

To reach our objective, we first give the following useful approximation lemma, which generalizes the corresponding result of Lepeltier and San Martin [10].

Lemma. Let $f : \Omega \times [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ be a measurable function such that:

For a.s. every $(t, \omega) \in [0,T] \times \Omega$, $f(t, \omega, x, y, z, \acute{y})$ is a continuous.

For every $(t, \omega, x, y, z, \acute{y}) \in [0,T] \times \Omega \times \mathbb{R}^4$

$$\begin{aligned} & |f(t, \omega, x, y, z, \acute{y})| \\ \le \; & \varphi(t) + r(t)(|x| + |y| + |\acute{y}|) + \theta(t)|z|. \end{aligned}$$

where φ, r and θ are three nonnegative processes such that for $a.e.$ $t \in [0,T]$, $\varphi(t)$, $r(t)$ and $\theta(t)$ $\mathcal{F}_t-$measurable.

Then exists the sequence of fonction f_n

$$\begin{aligned} & f_n(t, \omega, x, y, z, \acute{y}) \\ = \; & \inf_{(\tilde{y}, \tilde{z}, \tilde{y}^{'}, \tilde{z}^{'}) \in \mathbb{Q}} (f\left(t, \omega, x, \tilde{y}, \tilde{z}, \tilde{y}^{'}\right) \\ & + n\left(r(t)\left(|y - \tilde{y}| + \left|\acute{y} - \tilde{y}^{'}\right|\right) + \theta(t)|z - \tilde{z}|\right)), \end{aligned}$$

are well defined for $n \ge 1$ and satisfy the following conditions

(i) For all $n \ge 1$, $(t, \omega, x, y, z, \acute{y}) \in [0,T] \times \Omega \times \mathbb{R}^5$,

$$\begin{aligned} & |f_n(t, \omega, x, y, z, \acute{y})| \\ \le \; & \varphi(t) + r(t)(|x| + |y| + |\acute{y}|) + \theta(t)|z|. \end{aligned}$$

(ii) For any $(t,\omega,x,y,z,\acute{y})$, $f_n(t,\omega,x,y,z,\acute{y})$ is non-decreasing in n.

(iii) For all $(t,\omega,x,y,z,\acute{y}) \in [0,T]\times\Omega\times\mathbb{R}^4$, if $(t,\omega,x,y_n,z_n,\acute{y}_n) \to (t,\omega,x,y,z,\acute{y})$, then

$$f_n(t,\omega,x,y_n,z_n,\acute{y}_n) \to f(t,\omega,x,y,z,\acute{y}).$$

(iv) For any $n \geq 1$, $(t,\omega)\in[0,T]\times\Omega$, for all $(t,\omega,x,y,z,\acute{y}) \in [0,T]\times\Omega\times\mathbb{R}^4$ and $\left(t,\omega,x,\tilde{y},\tilde{z},\tilde{\acute{y}}\right) \in [0,T]\times\Omega\times\mathbb{R}^4$, we have

$$\begin{aligned} &\left|f_n(t,\omega,x,y,z,\acute{y}) - f_n\left(t,\omega,x,\tilde{y},\tilde{z},\tilde{\acute{y}}\right)\right| \\ \leq\ & n\left(r(t)\left(|y-\tilde{y}| + \left|\acute{y}-\tilde{\acute{y}}\right|\right) + \theta(t)|z-\tilde{z}|\right). \end{aligned}$$

3.1 Existence Result

Now, by the approximation method of the function f (**previous lemma**) and the comparison theorem, we establish the following existence theorem.

Theorem. *Assume **(A1)-(A3)**. Then, for δ sufficiently large, the MF-BSDE* (2) *has a minimal solution* $\left(\underline{Y},\underline{Z}\right) \in \tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}$.

Proof. We only prove that fractional MF-BSDE (2) has a minimal solution. Since $(A1)$ holds, it follows from **previous lemma** that there exists a sequence of stochastic Lipschitz-continuous functions f_n associated with f, which is non-decreasing in n.

Since

$$g(t) = \varphi(t) + r(t)(|x|+|y|+|\acute{y}|) + \theta(t)|z|$$

is stochastic Lipschitz, the existence and uniqueness result in the previous section implies that there exists a unique solution $(U,V) \in \tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}$ for fractional MF-BSDEs with data(ξ,g).

Now, for any $n\geq 1$, let a_n and A_n be two random processes with positive values defined by

$$\begin{aligned} a_n^2(t) &= nr(t) + n^2\theta^2(t) > 0, \text{ and} \\ A_n(t) &= \int_0^t a_n^2(s)\,ds < \infty. \end{aligned}$$

Then, in view of $(A1)-(A2)$, $a_n(t)$ and $A_n(t)$ are $\mathcal{F}_t-$measurable, for a.e. $t\in[0,T]$ such that for any $n\geq 1$, $0<a<a_n$ and $A<A_n<n^2A$. Thus, it is clair to deduce that for any $n\geq1$, $\tilde{\mathcal{V}}^{\frac{1}{2},a_n}_{[0,T]} \subset \tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]}$. Moreover, from $(A3)$, (ξ,f_n) satisfies the following conditions for any $n\geq1$:

$$\begin{aligned} \mathbb{E}\left(e^{\delta A_n(T)}|\xi|^2\right) &\leq \mathbb{E}\left(e^{\delta n^2 A(T)}|\xi|^2\right) \\ &< \infty, \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\int_0^T e^{\delta A_n(t)}\frac{|f_n(t,\omega,x,0,0,0)|^2}{a_n^2(t)}dt \\ \leq\ & \int_0^T e^{\delta n^2 A(t)}\frac{|\varphi(t)|^2}{a^2(t)}dt < \infty. \end{aligned}$$

Therefore, we get again from the previous section that for every $n\geq 1$ there exists a unique solution $(Y^n,Z^n)\in\tilde{\mathcal{V}}^{\frac{1}{2},a_n}_{[0,T]}\times\tilde{\mathcal{V}}^{H}_{[0,T]}$ for fractional MF-BSDE with data:

$$\begin{cases} -dY_t^n = \mathbb{E}'\left(f_n(t,\eta_t,Y_t^n,Z_t^n,\acute{Y}_t^n)\right)dt - Z_t^n dB_t^H, \\ Y_T^n = \xi,\ 0\leq t\leq T. \end{cases} \tag{19}$$

Consequently, for any $n\geq1$, $(Y^n,Z^n)\in\tilde{\mathcal{V}}^{\frac{1}{2},a}_{[0,T]}\times\tilde{\mathcal{V}}^{H}_{[0,T]}$.

On the other hand, since for fixed $(t,\omega,x,y,z,\acute{y})$ and all $n\geq1$,

$$\begin{aligned} f_n(t,\omega,x,y,z,\acute{y}) &\leq f_{n+1}(t,\omega,x,y,z,\acute{y}) \\ &\leq \varphi(t) + r(t)(|x|+|y|+|\acute{y}|) + \theta(t)|z|, \end{aligned}$$

it follows from the comparison theorem that for every $n\geq1$,

$$Y^n \leq Y^{n+1} \leq U,\ d\mathbb{P}\otimes dt - a.s. \tag{20}$$

The idea of the proof is to establish that the limit of the sequence (Y^n,Z^n) is a solution of the fractional MF-BSDE (2). To this end, we will sketch the proof in four steps.

Step 1: A priori estimates. There exists a constant $C>0$ independent of n such that

$$\begin{aligned} &\mathbb{E}e^{\delta A(t)}|Y_t^n|^2 + \mathbb{E}\int_t^T e^{\delta A(s)}a^2(s)|Y_s^n|^2 ds \\ &+\mathbb{E}\int_t^T e^{\delta A(s)}s^{2H-1}|Z_s^n|^2 ds \\ \leq\ & C, \end{aligned} \tag{21}$$

where C is a positive constant which may be different from line to line.

Indeed, for any $\delta>0$, Itô's formula applied to $e^{\delta A(t)}|Y_t^n|^2$ provides

$$\begin{aligned} &e^{\delta A(t)}|Y_t^n|^2 \\ =\ & e^{\delta A(T)}|\xi|^2 + 2\int_t^T e^{\delta A(s)}Y_s^n \\ &\times\mathbb{E}'\left(f_n(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n)\right)ds \\ &-2\int_t^T e^{\delta A(s)}Y_s^n Z_s^n dB_s^H - 2\int_t^T e^{\delta A(s)}\mathcal{D}_s^{\mathcal{H}}Y_s^n Z_s^n ds \\ &-\delta\int_t^T e^{\delta A(s)}|Y_s^n|^2 ds. \end{aligned}$$

Taking expectation, we get

$$\begin{aligned} &\mathbb{E}e^{\delta A(t)}|Y_t^n|^2 + \delta\mathbb{E}\int_t^T a^2(s)e^{\delta A(s)}|Y_s^n|^2 ds \\ &+2\mathbb{E}\int_t^T e^{\delta A(s)}\mathcal{D}_s^{\mathcal{H}}Y_s^n Z_s^n ds \\ =\ & \mathbb{E}e^{\delta A(T)}|\xi|^2 + 2\mathbb{E}\int_t^T e^{\delta A(s)}Y_s^n \\ &\times\mathbb{E}'\left(f_n(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n)\right)ds. \end{aligned}$$

It is known (see example Hu and Peng, [9], Maticiuc and Nie, [11]) that $\mathcal{D}_t^{\mathcal{H}}Y_t^n = (\hat{\sigma}(t)/\sigma(t))Z_t^n$. Moreover by Remark 6 in Maticiuc and Nie [11], there exists $M>0$ such that for all $t\in[0,T]$, $t^{2H-1}/M \leq \hat{\sigma}(t)/\sigma(t) \leq Mt^{2H-1}$.

Then, we have

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|Y_t^n\right|^2+\delta\mathbb{E}\int_t^T a^2(s)\,e^{\delta A(s)}\left|Y_s^n\right|^2ds\\
&+\frac{2}{M}\mathbb{E}\int_t^T s^{2H-1}e^{\delta A(s)}\left|Z_s^n\right|^2ds\\
\leq\;&\mathbb{E}e^{\delta A(T)}\left|\xi\right|^2+2\mathbb{E}\int_t^T e^{\delta A(s)}Y_s^n\\
&\times\mathbb{E}'\left(f_n(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n)\right)ds, \qquad (22)
\end{aligned}$$

assumption (A2) and property (iv) from previous lemma together with Young's inequality imply,

$$\begin{aligned}
&2Y_s^n\mathbb{E}'\left(f_n(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n)\right)\\
\leq\;&\left(5+\frac{M}{s^{2H-1}}\right)a^2(s)\left|Y_s^n\right|^2\\
&+\frac{s^{2H-1}}{M}\left|Z_s^n\right|^2+\frac{\left|\varphi(s)\right|^2}{a^2(s)}.
\end{aligned}$$

Therefore, for sufficiently large $\delta>0$, choosing δ such that $\left(\delta-5-\frac{M}{s^{2H-1}}\right)>1$, we obtain for $M>1$

$$\begin{aligned}
&\mathbb{E}\int_t^T a^2(s)\,e^{\delta A(s)}\left|Y_s^n\right|^2ds\\
&+\mathbb{E}\int_t^T s^{2H-1}e^{\delta A(s)}\left|Z_s^n\right|^2ds\\
\leq\;&C\mathbb{E}\left(e^{\delta A(T)}\left|\xi\right|^2+\int_t^T e^{\delta A(s)}\frac{\left|\varphi(s)\right|^2}{a^2(s)}ds\right)\\
<\;&\infty.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\mathbb{E}e^{\delta A(t)}\left|Y_t^n\right|^2\\
&+\mathbb{E}\int_0^T a^2(s)\,e^{\delta A(s)}\left|Y_s^n\right|^2ds\\
&+\mathbb{E}\int_0^T s^{2H-1}e^{\delta A(s)}\left|Z_s^n\right|^2ds\\
\leq\;&C\mathbb{E}\left(e^{\delta A(T)}\left|\xi\right|^2+\int_0^T e^{\delta A(s)}\frac{\left|\varphi(s)\right|^2}{a^2(s)}ds\right)\\
<\;&\infty.
\end{aligned}$$

Step 2: Convergence result. From (20) and (21), there exists a process Y^n such that $Y_t^n\nearrow Y_t$ a.s. for all $t\in[0,T]$. Therefore, it follows from Fatou's lemma together with the dominated convergence theorem that

$$\left\{\begin{array}{l}\mathbb{E}\int_0^T e^{\delta A(s)}\left|Y_s\right|^2ds\leq C,\text{ and}\\ \lim_{n\to\infty}\int_0^T a^2(s)\,e^{\delta A(s)}\left|Y_s^n-Y_s\right|^2ds=0.\end{array}\right. \qquad (23)$$

Next, for all $n\geq 1$, by Itô's formula applied to $e^{\delta A(t)}\left|Y_t^{n+1}-Y_t^n\right|^2$, we get

$$\begin{aligned}
&e^{\delta A(t)}\left|Y_t^{n+1}-Y_t^n\right|^2\\
=\;&2\int_t^T e^{\delta A(s)}\left(Y_s^{n+1}-Y_s^n\right)\\
&\times(\mathbb{E}'f_{n+1}\left(s,\eta_s,Y_s^{n+1},Z_s^{n+1},\acute{Y}_s^{n+1}\right)\\
&-\mathbb{E}'f_n\left(s,\eta_s,Y_s^n,Z_s^n,\acute{Y}_s^n\right))ds\\
&-2\int_t^T e^{\delta A(s)}\left(Y_s^{n+1}-Y_s^n\right)\left(Z_s^{n+1}-Z_s^n\right)dB_s^H\\
&-2\int_t^T e^{\delta A(s)}\mathcal{D}_s^{\mathcal{H}}\left(Y_s^{n+1}-Y_s^n\right)\left(Z_s^{n+1}-Z_s^n\right)ds\\
&-\delta\int_t^T e^{\delta A(s)}\left|Y_s^{n+1}-Y_s^n\right|^2ds.
\end{aligned}$$

Letting $t=0$, it follows from the uniform linear growth condition on the sequence f_n, (property (ii) in previous lemma), Cauchy-Schawrtz inequality and assumption **(A2)** that

$$\begin{aligned}
&\mathbb{E}\left|Y_0^{n+1}-Y_0^n\right|^2\\
&+\delta\mathbb{E}\int_0^T e^{\delta A(s)}a^2(s)\left|Y_s^{n+1}-Y_s^n\right|^2ds\\
&+\frac{2}{M}\mathbb{E}\int_0^T e^{\delta A(s)}s^{2H-1}\left|Z_s^{n+1}-Z_s^n\right|^2ds\\
\leq\;&C\mathbb{E}\int_0^T e^{\delta A(s)}(\frac{\left|\varphi(s)\right|^2}{a^2(s)}\\
&+a^2(s)\left(\left|Y_s^{n+1}\right|^2+\left|Y_s^n\right|^2+\left|\eta_s\right|^2\right)\\
&+\left(\left|Z_s^{n+1}\right|^2+\left|Z_s^n\right|^2\right)ds)^{\frac{1}{2}} \qquad (24)\\
&\times\left(\mathbb{E}\int_0^T e^{\delta A(s)}\left|Y_s^{n+1}-Y_s^n\right|^2ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, from (20) and assumption $(A3)$, we provide the existence of a constant $\tilde{C}>0$ independent of n such that

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{\delta A(s)}s^{2H-1}\left|Z_s^{n+1}-Z_s^n\right|^2ds\\
\leq\;&\tilde{C}\left(\mathbb{E}\int_0^T e^{\delta A(s)}\left|Y_s^{n+1}-Y_s^n\right|^2ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Consequently, it follows from (23) that $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $\tilde{\mathcal{V}}_{[0,T]}^H$. Then there exists an $\mathcal{F}_t-$jointly measurable process $Z\in\tilde{\mathcal{V}}_{[0,T]}^H$ such that

$$\lim_{n\to\infty}\mathbb{E}\int_0^T e^{\delta A(s)}s^{2H-1}\left|Z_s^n-Z_s\right|^2ds=0.$$

Step 3: (Y,Z) verifies MF-BSDE driven by fBm (2). Since $(Y^n,Z^n)\to(Y,Z)$ in $\tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a}\times\tilde{\mathcal{V}}_{[0,T]}^H$, along a subsequence which we still denote (Y^n,Z^n), we get

$$(Y^n,Z^n)\to(Y,Z)\;\;dt\otimes d\mathbb{P}\;a.e.,$$

and there exists $\chi \in \tilde{\mathcal{V}}_{[0,T]}^{H}$ such that for all $n \geq 1$, $|Z^n| < \chi \; dt \otimes d\mathbb{P}$ *a.e.* Therefore, by the previous lemmma, we have

$$\begin{aligned} & f_n\left(t, \eta_t, Y_t^n, Z_t^n, \acute{Y}_t^n\right) \\ \rightarrow \; & f\left(t, \eta_t, Y_t, Z_t, \acute{Y}_t\right) \; dt \otimes d\mathbb{P} \; a.e., \end{aligned}$$

Moreover, from condition (ii) in the previous lemma and (20), we have

$$\begin{aligned} & \left| f_n\left(t, \eta_t, Y_t^n, Z_t^n, \acute{Y}_t^n\right) \right| \\ \leq \; & \Sigma(t) < \infty \; dt \otimes d\mathbb{P} \; a.e., \end{aligned}$$

where

$$\begin{aligned} \Sigma(t) \;=\; & \varphi(t) + r(t)\left(\left|Y_t^1\right| + \left|\acute{Y}_t^1\right| + |U_t|\right) \\ & + \theta(t)\,|\chi_t| \,. \end{aligned}$$

Then it follows from the dominated convergence theorem that

$$\begin{aligned} & \mathbb{E}\int_t^T f_n\left(s, \eta_s, Y_s^n, Z_s^n, \acute{Y}_s^n\right) ds \\ \underset{n\to\infty}{\rightarrow} \; & \mathbb{E}\int_t^T f\left(s, \eta_s, Y_s, Z_s, \acute{Y}_s\right) ds. \end{aligned}$$

Finally, passing to the limit on both sides of fractional MF-BSDE (19), we get that (Y, Z) is a solution of MF-BSDE (2) .

Step 4: Minimal solution. Let $\left(\tilde{Y}, \tilde{Z}\right) \in \tilde{\mathcal{V}}_{[0,T]}^{\frac{1}{2},a} \times \tilde{\mathcal{V}}_{[0,T]}^{H}$ be any solution of fractional MF-BSDE (2) and let us consider for any $n \geq 1$ the fractional MF-BSDE (19) with its unique solution (Y^n, Z^n), which converges to (Y, Z). Since $f_n \leq f$ for all $n \geq 1$, we get by virtue of the comparison theorem that $Y^n \leq \tilde{Y}$ for all $n \geq 1$. Therefore, $Y \leq \tilde{Y}$.

That proves that (Y, Z) is the minimal solution for fractional MF-BSDE (2). □

4 Conclusion

In the first part of this work, we have studied mean field backward stochastic differential equations driven by fBm with Hurst parameter $H > \frac{1}{2}$ under stochastic Lipschitz condition. In the second part of the paper we establish the existence of minimal solution to the mean field backward stochastic differential equations driven by fBm with Hurst parameter $H > \frac{1}{2}$ under continuous and stochastic linear growth condition. Motivated by the works of [3, 4, 9, 10, 14, 13], we have proved an existence result to this kind of equations, in which the coefficient f is assumed to be continuous and stochastic linear growth condition, more precisely, we have treated the stochastic Lipschitz case. So our method in continuous and stochastic linear growth condition case is similar techniques developed in [10] with some suitable changes due to the difference between the processes and the spaces. We note that pretty much of the technical difficulties coming from the fractional brownien motion, since B^H with $H > \frac{1}{2}$ is not a semimartingale, we cannot use the classical theory of stochastic calculus.

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