ISSN: 1803-3814 (Printed), 2571-3701 (Online)
https://doi.org/10.13164/mendel.2021.2.030

MENDEL Soft Computing Journal

Significant Curves of the Mandelbrot Set

Dalibor Martišek

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Brno, Czech Republic
martisek@fme.vutbr.cz

Abstract

The paper provides a description of some interesant curves contained in the Mandelbrot set. These curves are the boundaries of the areas called bulbs" which are described approximately only in present. In this paper, some of them are described analyticaly – curves of so called first period, the boundary of the main hyperbolic component, internal and external bounds and also some curves of the second period.

Keywords: *Mandelbrot Set, Main Hyperbolic Component, Internal Bound, External Bound, Resultant of Polynomials.*

Received: 02 October 2021
Accepted: 03 December 2021
Published: 21 December 2021

1 Introduction

The Mandelbrot set was discovered in 1979 during an attempt to castalogize the sets that Gaston Julia and Piere Fatau studied in the second decade of the 20th century. It is a dynamic system whose behavior is described by the equation

$$z_0 = 0;\ z_{n+1} = z_n^2 + c; n \in \mathbb{N};\ z_n; c \in \mathbb{C} \tag{1}$$

or more generally

$$z_0 = 0;\ z_{n+1} = f(z_n; c); n \in \mathbb{N};\ z_n; c \in \mathbb{C} \tag{2}$$

where $f(z)$ is a holomorfic function. This series converges for some values of c and diverges for others points. The Mandelbrot set consists of all points $c \in \mathbb{C}$, for which the process (1) or (2) does not diverge. Formally:

$$\mathcal{M}(f) = \left\{ c \in \mathbb{C} | \lim_{n\to\infty} f(z_n; c) \neq \infty; n \in \mathbb{N}; z_n; c \in \mathbb{C} \right\} \tag{3}$$

2 Visual Representation of the Mandelbrot Set

The visual representation of the Mandelbrot set may be created by determining, for each point $C \in \mathbb{C}$ of a part of the complex plane, whether z_n is bounded. The number of iterations to reach a chosen radius can be used to determine the color to use – it is so called the Integer Escape-Time (IET) algorithm [3, 1, 5]. However, this algorithm creates clearly visible colour discontinuities – see the upper part of Fig. 1. The so-called Smooth Escape-Time algorithm is more fitting for visualizations. It is applicable to polynomial function $f(z) = z^k + c_{k-1}z^{k-1} + \ldots + c_1 z + c_0 \approx z^k + c_0$ (because $|z| \gg 1$ therefore $|z^k| \gg |z^{k-1}|$). Let us assume that r is chosen escape radius and the divergence is detected in the n-th step. In this case, $|z_n| \in \left(r; r^k\right)$ and (constant) n-th colour belongs to this whole interval in IET algorithm. It is necessary to calculate a parameter $h \in (0; 1)$ to assign "n+h"-th colour for continuous transition. We need a suitable function for the transformation $\left(r; r^k\right) \to (0; 1)$. Value $h = \log_k \log_r |z_n|$ is applicable for this purpose for example – see the lower part of Fig. 1. In the following text, we will analyse sets $\mathcal{M}_k(f)$ – sets according to (3) for which

$$f(z_n; c) = z_{n+1} = z_n^k + c. \tag{4}$$

3 Curves of the first period

Curves of the first period are sets

$$\mathcal{C}_1 = \left\{ z \in \mathbb{C} \,\middle|\, z = z^k + c \right\}.$$

3.1 The Boundary of the Main Hyperbolic Component

The main hyperbolic component $\mathcal{H}_k^{(1)}$ is the subset of $\mathcal{M}_k(f(z))$ for which the orbit does not diverge [2, 4]. It is bounded by $\mathcal{C}_1$. In case of (4),

$$f(z) = z^k + c \tag{5}$$

and equation of $\mathcal{C}_1$ is possible to obtain from $f(z) = z = z^k + c$, i.e.,

$$c = z - z^k. \tag{6}$$

According to Banach fixed-point theorem, its points must satisfy the condition $|f'(z)| \leq 1$ and for its boundary $|f'(z)| = 1$. It means that $f'(z) = e^{i\omega}$ in the Gaussian plane. Therefore, if we assign

$$\begin{aligned} z &= \sqrt[k-1]{\frac{1}{k}\exp(i\omega)} = \left[\frac{1}{k}\exp(i\omega)\right]^{\frac{1}{k-1}} \\ &= \frac{1}{k^{\frac{1}{k-1}}}\exp\left(\frac{i\omega}{k-1}\right) \end{aligned} \tag{7}$$

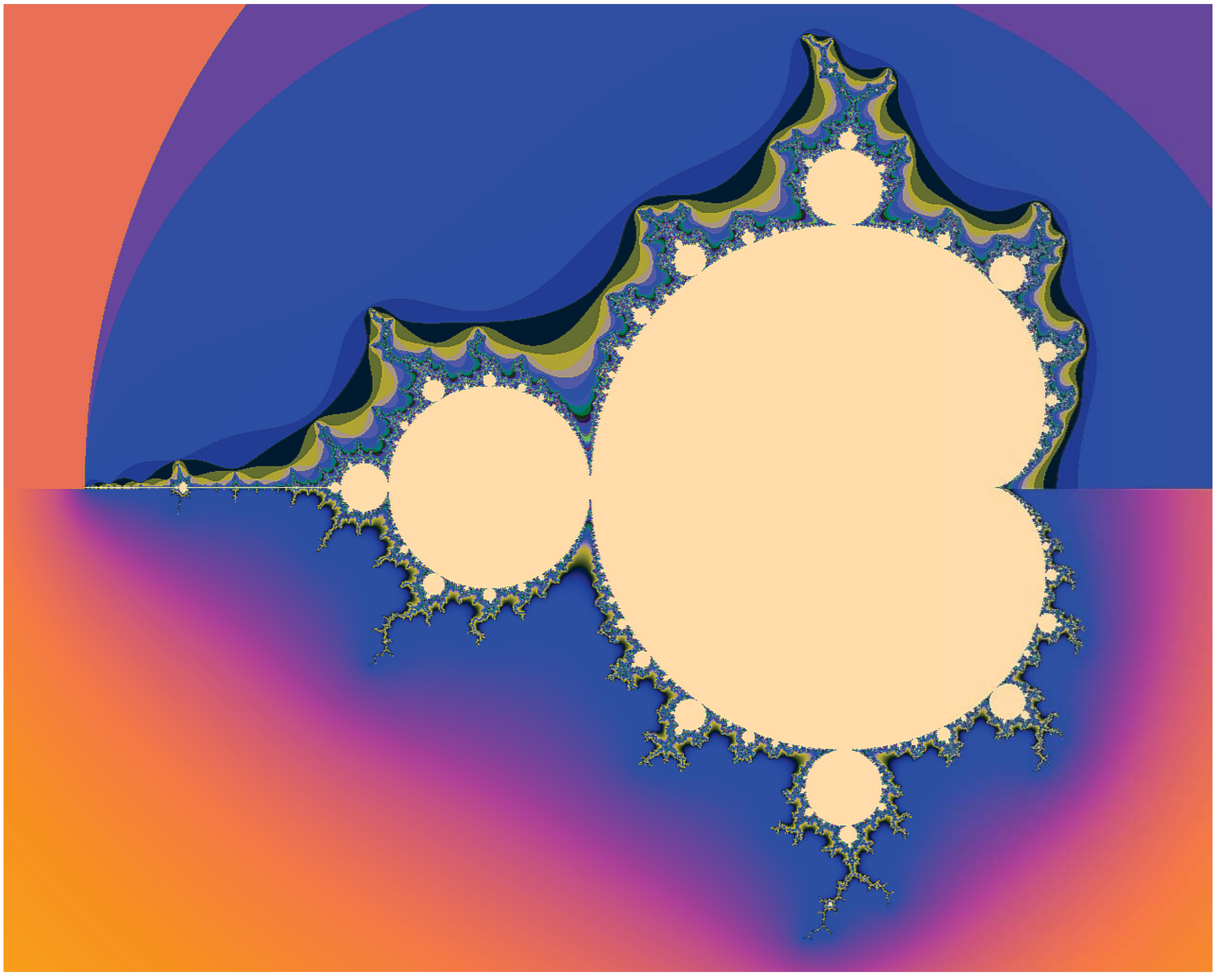

Figure 1: Visualization of the Mandelbrot set. Integer Escape-Time (up), Smooth Escape-Time (down).

then

$$
\begin{aligned}
f'(z) &\overbrace{=}^{(5)} (z^k + C)' = k \cdot z^{k-1} \\
&= k \cdot \left[\frac{1}{k^{\frac{1}{k-1}}} \exp\left(\frac{i\omega}{k-1} \right) \right]^{k-1} = \exp(i\omega)
\end{aligned}
\tag{8}
$$

and

$$
c \overbrace{=}^{(6)} z - z^k \overbrace{=}^{(7)} \frac{1}{k^{\frac{1}{k-1}}} \exp\left(\frac{i\omega}{k-1} \right) - \frac{1}{k^{\frac{k}{k-1}}} \exp\left(\frac{i\omega}{k-1} \right)
\tag{9}
$$

For simplification, let us denote $\frac{i\omega}{k-1} = \varphi$ and we obtain boundaries of the main hyperbolic components as

$$
c_k^{(1)} = \frac{1}{k^{\frac{1}{k-1}}} \cdot \exp(i\varphi) - \frac{1}{k^{\frac{k}{k-1}}} \cdot \exp(i\varphi k); \quad 0 \leq \ \varphi < \pi
\tag{10}
$$

The subscript (1) means that this is the curve of period one – if z_n lies on this curve, then also z_{n+1} lies on the curve too. For the classic Mandelbrot set $z_{n+1} = z_n^2 + C$ is $k = 2$ and we have

$$
\begin{aligned}
c_2^{(1)}(\varphi) &= \frac{1}{2} e^{i\varphi} - \frac{1}{4} e^{2i\varphi} \\
&= \frac{1}{2}(\cos\varphi + i \cdot \sin\varphi\ ) - \frac{1}{4}(\cos 2\varphi + i \cdot \sin 2\varphi)
\end{aligned}
$$

For $z_{n+1} = z_n^3 + C$ is $k = 3$ and

$$
\begin{aligned}
c_3^{(1)}(\varphi) &= \frac{1}{\sqrt{3}} \left(e^{i\varphi} - \frac{1}{3} e^{3i\varphi} \right) \\
&= \frac{1}{\sqrt{3}}(\cos\varphi + i \cdot \sin\varphi) - \frac{\sqrt{3}}{9}(\cos 3\varphi + i \cdot \sin 3\varphi)
\end{aligned}
$$

By analogy

$$
\begin{aligned}
c_4^{(1)}(\varphi) &= \frac{1}{\sqrt[3]{4}} \left(e^{i\varphi} - \frac{1}{4} e^{4i\varphi} \right) \\
&= \frac{1}{\sqrt[3]{4}}(\cos\varphi + i \cdot \sin\varphi) - \frac{1}{4\sqrt[3]{4}}(\cos 4\varphi + i \cdot \sin 4\varphi)
\end{aligned}
$$

These curves are marked as pink on Figures 2, 3, and 4. It is important to note that $\lim_{k\to\infty} \frac{1}{k^{\frac{1}{k-1}}} = 1$ and $\lim_{k\to\infty} \frac{1}{k^{\frac{k}{k-1}}} = 0$ in (10) and therefore

$$
\lim_{k\to\infty} c_k^{(1)}(\varphi) = e^{i\varphi},
$$

which means that these curves converge into the unit circle.

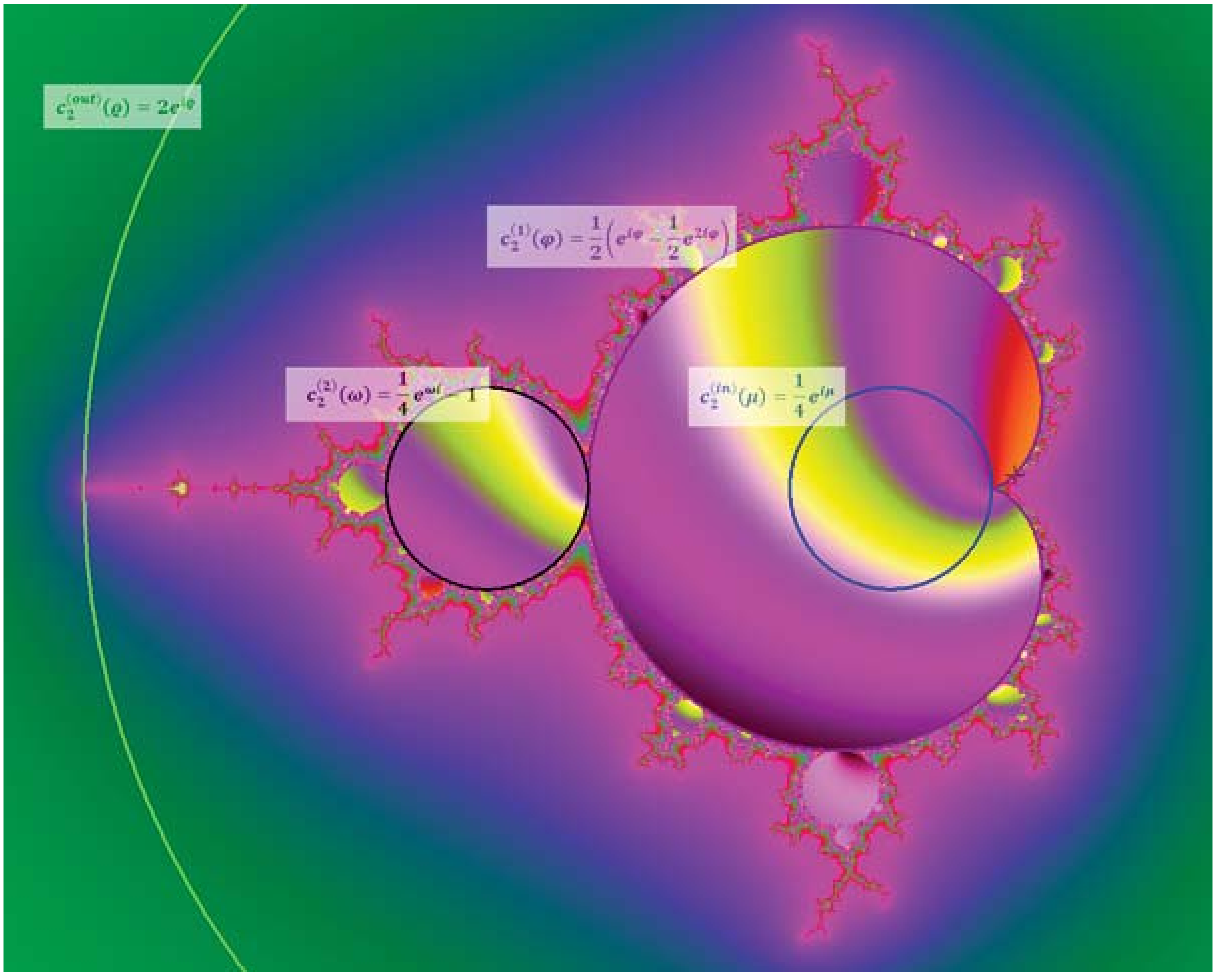

Figure 2: Curves of the first and second period for $k = 2$.

3.2 Internal Bound

For speeding up rendering of the Mandelbrot set, it is possible to detect as internal points those belonging into the curve of period one. They lie on the circle with the centre $S = 0$ and radius $r = \min\left\{\left|c_k^{(1)}\right|\right\}$. Because

$$
\begin{aligned}
\left|c_k^{(1)}\right|^2 &= \left(\frac{1}{k^{\frac{1}{k-1}}}\cos\varphi - \frac{1}{k^{\frac{k}{k-1}}}\cos k\varphi\right)^2 \\
&\quad + \left(\frac{1}{k^{\frac{1}{k-1}}}\sin\varphi - \frac{1}{k^{\frac{k}{k-1}}}\sin k\varphi\right)^2 \\
&= \frac{1}{k^{\frac{2}{k-1}}}\cos^2\varphi - 2\frac{1}{k^{\frac{1}{k-1}}}\cos\varphi\frac{1}{k^{\frac{k}{k-1}}}\cos k\varphi \\
&\quad + \frac{1}{k^{\frac{2k}{k-1}}}\cos^2 k\varphi + \frac{1}{k^{\frac{2}{k-1}}}\sin^2\varphi \\
&\quad - 2\frac{1}{k^{\frac{1}{k-1}}}\sin\varphi\frac{1}{k^{\frac{k}{k-1}}}\sin k\varphi + \frac{1}{k^{\frac{2k}{k-1}}}\sin^2 k\varphi \\
&= \frac{1}{k^{\frac{2}{k-1}}} \\
&\quad - 2\frac{1}{k^{\frac{1}{k-1}}}\frac{1}{k^{\frac{k}{k-1}}}\left(\cos\varphi\cos k\varphi + \sin\varphi\sin k\varphi\right) \\
&\quad + \frac{1}{k^{\frac{2k}{k-1}}}
\end{aligned}
$$

$$
\left|c_k^{(1)}\right|^2 = \frac{1}{k^{\frac{2}{k-1}}} - 2\frac{1}{k^{\frac{1}{k-1}}}\frac{1}{k^{\frac{k}{k-1}}}\cos\left[(k-1)\varphi\right] + \frac{1}{k^{\frac{2k}{k-1}}}
$$

for which the minimum occurs for $\cos\left[(k-1)\varphi\right] = 1$ (i.e. $\varphi = 0$), therefore

$$
\begin{aligned}
\min\left\{\left|c_k^{(1)}\right|^2\right\} &= \left(\frac{1}{k^{\frac{1}{k-1}}}\right)^2 - 2\frac{1}{k^{\frac{1}{k-1}}}\frac{1}{k^{\frac{k}{k-1}}} \\
&= \left(k^{\frac{1}{1-k}} - k^{\frac{k}{1-k}}\right)^2
\end{aligned}
$$

and

$$
r = \min\left\{\left|c_k^{(1)}\right|\right\} = k^{\frac{1}{1-k}} - k^{\frac{k}{1-k}} \tag{11}
$$

They are circles with the center $S = 0$ and radius (14), its equations for $k = 2; 3; 4$ are

$$
c_2^{(in)}(\mu) = \frac{1}{4}e^{i\mu};\, c_3^{(in)}(\mu) = \frac{2\sqrt{3}}{9}e^{i\mu};\, c_4^{(in)}(\mu) = \frac{3\sqrt[3]{4}}{16}e^{i\mu}
$$

These circles are marked as pink in Figures 2, 3, 4.

3.3 External Bound

Rendering of the Mandelbrot set is possible to accelerate also by external bound detection – the escape

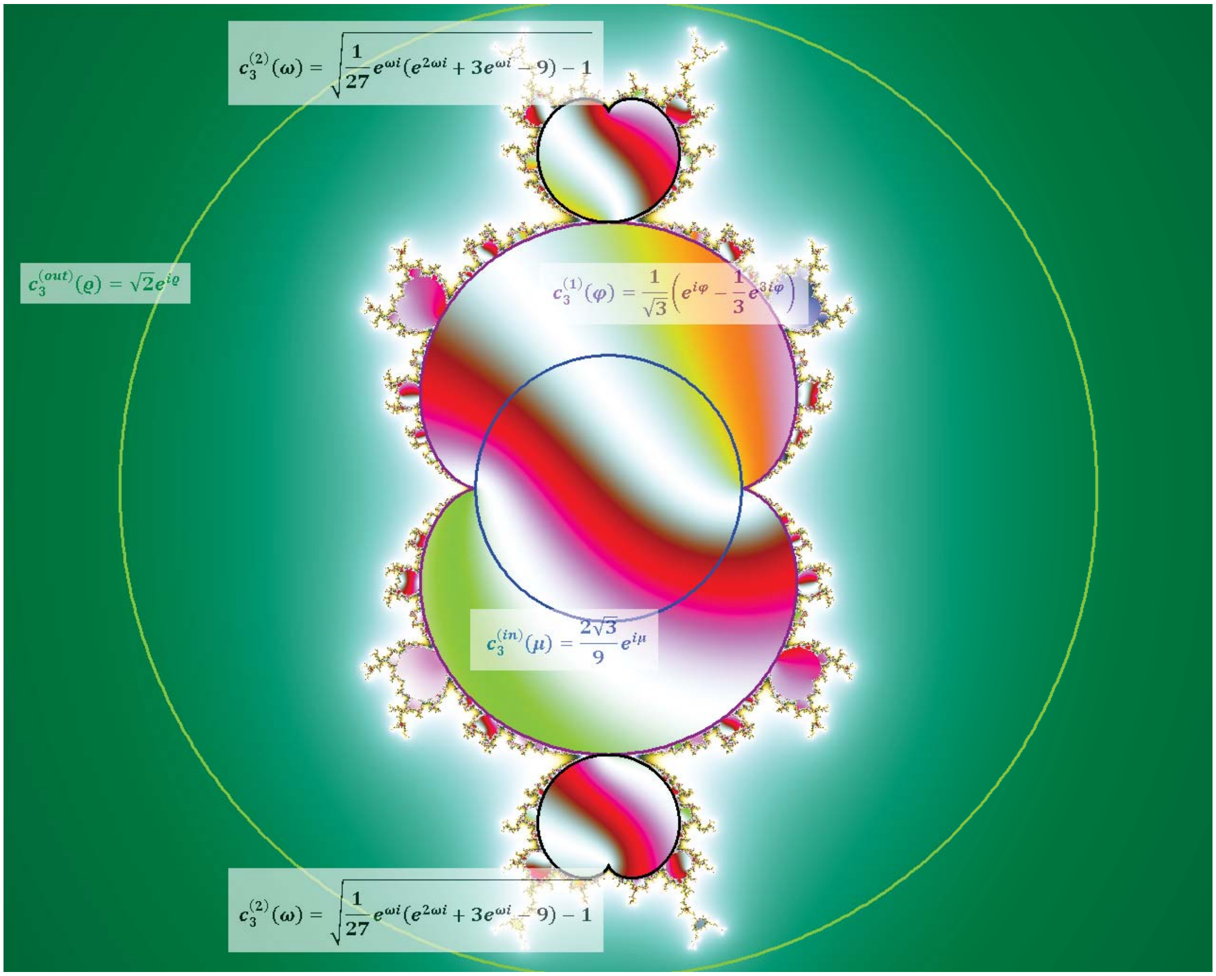

Figure 3: Curves of the first and second period for $k = 3$.

radius mentioned in section 2 does not have to be bigger than radius of the external bound. We can assume that outside the escape zone $|z_n|$ is already bigger than $|c|$ therefore

$$\frac{|z_{n+1}|}{|z_n|} = \frac{|z_n^k + c|}{|z_n|} > 1$$

Using the anti-triangle inequality $|a + b| \geq ||a| - |b||$ for complex numbers we get

$$\frac{|z_n^k + c|}{|z_n|} \geq \frac{\left||z_n|^k - |c|\right|}{|z_n|} = |z_n|^{k-1} - \frac{|c|}{|z_n|}$$

Since $|z_n| > |c|$, we have

$$\frac{|z_n^k + c|}{|z_n|} \geq |z_n|^{k-1} - \frac{|c|}{|z_n|} \geq |z_n|^{k-1} - 1 \geq 1 \Rightarrow$$

$$|z_n|^{k-1} \geq 2 \Rightarrow |z_n| \geq 2^{\frac{1}{k-1}}$$

Which means that the external bounds are the circles with centre $S = 0$ and radius $r = 2^{\frac{1}{k-1}}$, their equations for $k = 2; 3; 4$ are

$$c_2^{(out)}(\varrho) = 2e^{i\varrho}; c_3^{(out)}(\varrho) = \sqrt{2}e^{i\varrho}; c_4^{(out)}(\varrho) = \sqrt[3]{2}e^{i\varrho}$$

These circles are marked as pink in Figures 2, 3, 4.

4 Curves of the Second Period

Curves of the second period are sets

$$\mathcal{C}_2 = \left\{ z \in \mathbb{C} \,\middle|\, z = \left(z^k + c\right)^k + c \right\} - \mathcal{C}_1$$

These curves are boundaries between convergence and divergence process

$$z = \left(z^k + c\right)^k + c$$

According to Banach fixed-point theorem, the following relationship

$$|f'(z)| = \left|\left(\left(z^k + c\right)^k + c\right)'\right| = 1 \tag{12}$$

must hold again.

4.1 Curves of the second period for $k = 2$

We have to find roots of

$$z = \left(z^2 + c\right)^2 + c \quad \Rightarrow \quad z^4 + 2z^2c - z + c\,(c + 1) = 0$$

that are not roots of

$$z = z^2 + c \Rightarrow z^2 - z + c = 0$$

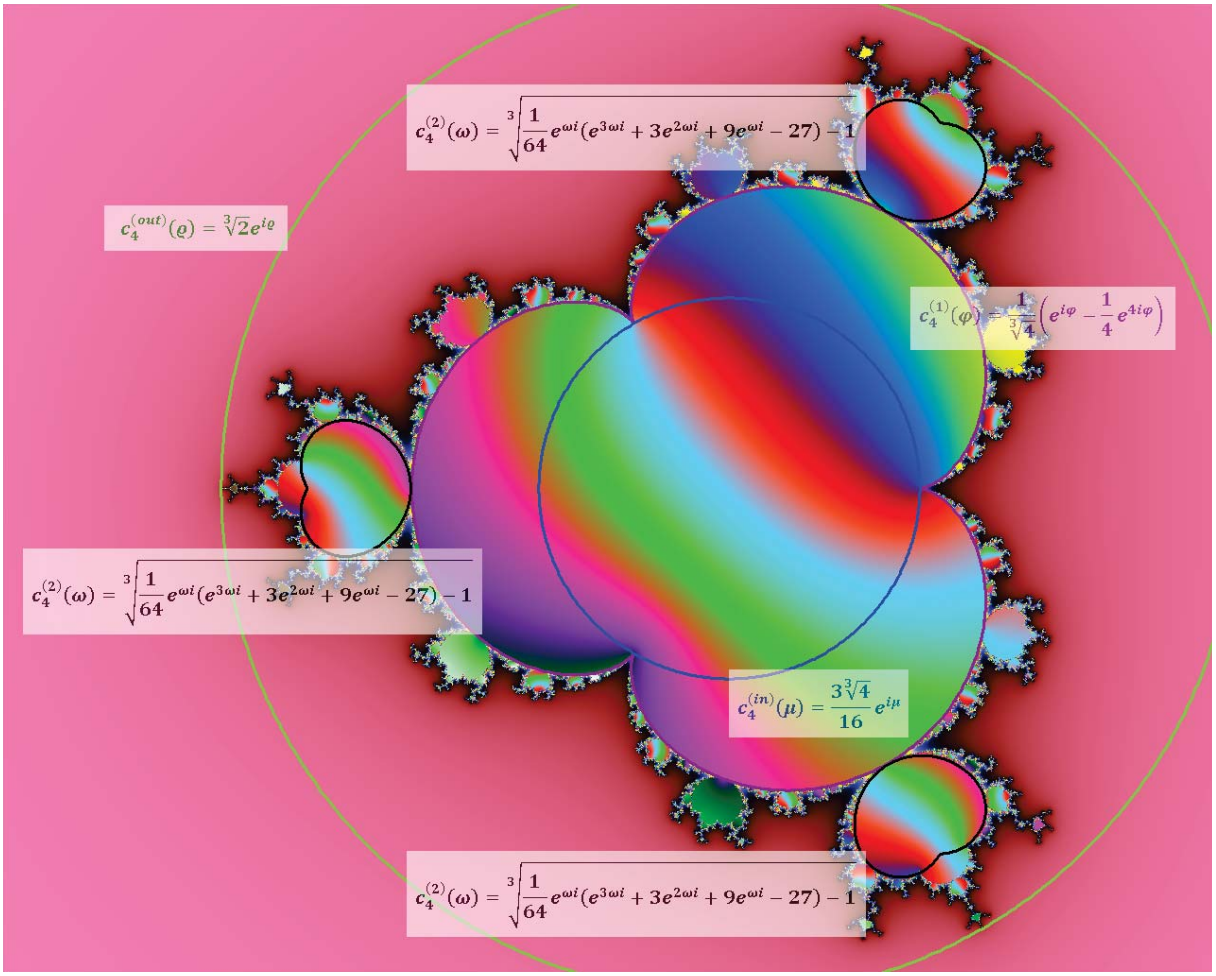

Figure 4: Curves of the first and second period for $k = 4$.

Quotient of these polynomials must be equal to zero

$$\begin{aligned} &\left(z^4 + 2z^2c - z + c\,(c+1)\right) : \left(z^2 - z + c\right) \\ &\quad = z^2 + z + c + 1 = 0 \end{aligned} \tag{13}$$

Equation (12) gives

$$\left|((z^2+c)^2+c)'\right| = \left|2\left(z^2+c\right)\cdot 2z\right| = \left|4z^3 + 4cz\right| = 1$$

Therefore

$$4z^3 + 4cz = e^{\omega i}$$

Using

$$\frac{1}{4}e^{\omega i} = \lambda \tag{14}$$

we obtain

$$z^3 + cz - \frac{1}{4}\lambda = 0 \Rightarrow \quad z^3 + cz - \lambda = 0 \tag{15}$$

Now, it is necessary to calculate the resultant of polynomials (13) (15), i.e., the expression of their coefficients, which is equal to zero if and only if the polynomials have a common root. Therefore, we must construct the determinant of the Sylvester matrix and set it equal to zero.

$$\begin{vmatrix} 1 & 1 & 1+c & 0 & 0 \\ 0 & 1 & 1 & 1+c & 0 \\ 0 & 0 & 1 & 1 & 1+c \\ 1 & 0 & c & -\lambda & 0 \\ 0 & 1 & 0 & c & -\lambda \end{vmatrix} = (1+c-\lambda)^2 = 0$$

We obtain $1 + c - \lambda = 0$ and from substitution (14)

$$c_2^{(2)}(\omega) = \frac{1}{4}e^{\omega i} - 1$$

This circle is marked as black in Figures 2, 3, 4.

4.2 Curves of the second period for $k = 3$ and 4

According to previous section: for $k = 3$, we have to find roots of

$$z = \left(z^3 + c\right)^3 + c \Rightarrow z^9 + 3cz^6 + 3c^2z^3 - z + \left(c^3 + c\right) = 0$$

that are not roots of

$$z = z^3 + c \quad \Rightarrow \quad z^3 - z + c = 0$$

Quotient

$$\left(z^9 + 3cz^6 + 3c^2z^3 - z + \left(c^3 + c\right)\right) : \left(z^3 - z + c\right) \\ = z^6 + z^4 + 2cz^3 + z^2 + cz + c^2 + 1 = 0 \tag{16}$$

Equation (12) gives

$$\left|\left(\left(z^3 + c\right)^3 + c\right)'\right| = \left|3\left(z^3 + c\right)^2 \cdot 3z^2\right| \\ = \left|9z^2\left(z^3 + c\right)^2\right| = 1$$

Therefore

$$9z^2\left(z^3 + c\right)^2 = e^{\omega i}$$

Using

$$\frac{1}{9}e^{\omega i} = \lambda \tag{17}$$

we obtain

$$z^8 + 2z^5c + c^2z^2 - \lambda = 0 \tag{18}$$

Determinant of Sylvester matrix of resultant of (16)-(18) is

$$\left(c^2 - (\lambda - 1)(\lambda + 1)^2\right)^2 = 0$$

and for c, we have

$$c_3^{(2)}(\omega) = \sqrt{\frac{1}{3}\lambda\left(\lambda^2 + 3\lambda - 9\right) - 1} \\ = \sqrt{\frac{1}{27}e^{\omega i}\left(e^{2\omega i} + 3e^{\omega i} - 9\right) - 1}$$

For $k = 4$, we can obtain

$$c_4^{(2)}(\omega) = \sqrt[3]{\frac{1}{4^4}\left(e^{4\omega i} + 4e^{3\omega i} + 16e^{2\omega i} - 64e^{\omega i}\right) - 1}$$

The calculation is already technically difficult, so we do not perform it.

5 Conclusion

In [4], we can read about the Mandelbrot set of the second degree: "It should be pointed out that the bulbs' apparent circular shape is indeed only approximate." In this paper, we proved, that the curve of the second period in this set is precise circle. The curves of the higher period cannot be precise circles because the Mandelbrot set is generated by nonlinear transforms. However, it is possible to obtain their analytical description on principle.

Acknowledgement: The author acknowledges support from Private Institute of Applied Mathematics, Slapanice, Czech Republic.

References

[1] Devaney, R. The fractal geometry of the mandelbrot set ii: How to add and how to count. *Fractals 03*, 04 (1995), 629–650.

[2] Devaney, R. Unveiling the mandelbrot set. *Plus Magazine 40* (2006).

[3] Douady, A., and Hubbard, J. H. Itération des polynômes quadratiques complexes (iteration of complex quadratic polynomials). *C. R. Acad. Sci., Paris, Sér. I 294* (1982), 123–125.

[4] Fowler, A., and McGuinness, M. The size of mandelbrot bulbs. *Chaos, Solitons & Fractals: X 3* (2019), 100019.

[5] Goldberg, L., and Tresser, C. Rotation orbits and the farey tree. *Ergodic Theory and Dynamical Systems 16*, 5 (1996), 1011–1029.